ESD-TDR-65-46

Group Report 1965-10

S. L. Borison

Statistics of the Radar Cross Section of a Volume of Chaff

10 February 1965

Prepared for the Advanced Research Projects Agency
under Electronic Systems Division Contract AF 19(628)-500 by

Lincoln Laboratory
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Lexington, Massachusetts

The work reported in this document was performed at Lincoln Laboratory, a center for research operated by Massachusetts Institute of Technology. This research is a part of Project DEFENDER, which is sponsored by the U.S. Advanced Research Projects Agency of the Department of Defense; it is supported by ARPA under Air Force Contract AF 19(628)-500 (ARPA Order 498).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

STATISTICS OF THE RADAR CROSS SECTION OF A VOLUME OF CHAFF

S. L. BORISON

Group 41

GROUP REPORT 1965-10

10 FEBRUARY 1965

LEXINGTON MASSACHUSETTS

Abstract

The average and standard deviation of the radar cross section of chaff is computed under the assumptions that dipoles are randomly oriented and randomly distributed within a radar resolution volume. For a single type of chaff, the standard deviation approaches the average value as the average number of dipoles increases. When the number of dipoles per resolution volume is small, the statistics of the single dipole cross section are important. The results are then generalized to the case of several types of dipoles distributed in space. The average cross section is simply the sum of the average cross sections for each type; however, the standard deviation involves additional terms which are not small. These terms are just sufficient to again provide the Rayleigh limit $\delta s / < s > \rightarrow 1$ as the number of dipoles increases.

Accepted for the Air Force
Stanley J. Wisniewski
Lt Colonel, USAF
Chief, Lincoln Laboratory Office

I. Introduction

In a previous report[1] the author considered the statistics of the radar cross section of a small number of resonant dipoles randomly oriented and randomly spaced in a radar resolution volume. In that case it was possible to use an accurate analytic representation of the cross section of a single dipole to determine a complete representation of the statistics of the radar cross section, i. e. , the probability density. However, it will more often be the case that the radar frequency will not correspond to the resonant frequency of the chaff. In fact, the chaff may be dispensed in several lengths some of which may be resonant to radar frequencies almost an octave lower than the operating frequency.

In order to understand this more complicated chaff environment we have calculated the average radar cross section and its standard deviation in terms of the average and standard deviation for the single dipole.[2-3] The average cross section is merely the summation of the individual average dipole cross sections. However, the standard deviation of the cross section is a good indication of the importance of the statistics of the single dipole cross section. In particular, it is to be expected that once the standard deviation, δs , is approximately equal to the average cross section, $\langle s \rangle$, the probability density for the cross section is approximately exponential (i. e. , Rayleigh power distribution). In section II, we have considered the chaff to be of a single type, and in section III we have considered the more general case of several types of chaff. In each case it is found that the Rayleigh limit is reached for large numbers of dipoles. If one assumes that the standard deviation of the

single dipole cross section is about equal to the average, the Rayleigh limit is well approximated by two or three dipoles. Furthermore, for the case of small numbers of dipoles, the results enable one to calculate the first and second order statistics accounting for the single dipole statistics.

II. Statistics For a Single Type of Chaff.

Let A_ℓ be the coherent amplitude for the ℓ-th radar resolution cell,

a_α be the coherent amplitude for the α-th dipole in the ℓ-th cell,

r_α be the range to the α-th dipole in the ℓ-th cell,

n_ℓ be the number of dipoles in the ℓ-th cell, and

$k = 2\pi/\lambda$ be the wave-number.

$$A_\ell = \sum_{\alpha=1}^{n_\ell} a_\alpha e^{i2kr_\alpha}$$

$$s_\ell = A_\ell A_\ell^* = \sum_{\alpha=1}^{n_\ell} \sum_{\beta=1}^{n_\ell} a_\alpha a_\beta^* \, e^{i2k(r_\alpha - r_\beta)} \quad .$$

For n_ℓ fixed, and all dipoles statistically independent, randomly oriented, and randomly spaced in the resolution volume Δv,

$$\langle s_\ell \rangle = \frac{1}{(\Delta v)^{n_\ell}} \int d\vec{r}_1 \ldots d\vec{r}_{n_\ell} \, d\Omega_1 \ldots d\Omega_{n_\ell} \sum_{\alpha,\beta}^{n_\ell} a_\alpha a_\alpha^* \, e^{i2k(r_\alpha - r_\beta)} \quad ,$$

where $d\vec{r}_i$ is the volume element and $d\Omega_i$ is the orientation surface element for the i-th dipole.

For $r_\alpha \neq r_\beta$ (i.e., $\alpha \neq \beta$) ,

$$\int d\vec{r}_\alpha e^{i2kr_\alpha} = \int d\vec{r}_\beta \, e^{i2kr_\beta} = 0 \quad ,$$

and

$$\langle \sigma_\ell \rangle = \int d\Omega_1 \ldots d\Omega_{n_\ell} \sum_\alpha^{n_\ell} a_\alpha a_\alpha^* \equiv \sum_\alpha^{n_\ell} \langle \sigma \rangle = n_\ell \langle \sigma \rangle \quad .$$

If n_ℓ is considered to be a random variable,

$$\boxed{\langle s_\ell \rangle = \langle n_\ell \rangle \langle \sigma \rangle} \quad . \qquad (1)$$

To determine the standard deviation of s_ℓ , we must calculate

$$\langle s_\ell^2 \rangle = \frac{1}{(\Delta v)^{n_\ell}} \int d\vec{r}_1 \ldots d\vec{r}_{n_\ell} d\Omega_1 \ldots d\Omega_{n_\ell} \sum_{\alpha,\beta,\gamma,\delta}^{n_\ell} a_\alpha a_\beta^* a_\gamma a_\delta^* e^{i2k(r_\alpha - r_\beta + r_\gamma - r_\delta)} \quad .$$

The cases for which the $d\vec{r}_i$ integrals are non-zero are

1) $\underline{\alpha = \beta = \gamma = \delta}$

2) $\underline{\alpha \neq \gamma}$

a) $\alpha = \beta\ , \quad \gamma = \delta$

b) $\alpha = \delta\ , \quad \beta = \gamma$.

Thus,

$$\langle s_\ell^2 \rangle = \int d\Omega_1 \ldots d\Omega_{n_\ell} \left[\sum_{\alpha}^{n_\ell} (a_\alpha a_\alpha^*)^2 + 2 \sum_{\alpha \neq \gamma}^{n_\ell} a_\alpha a_\alpha^* \, a_\gamma a_\gamma^* \right]$$

$$= \sum_{\alpha}^{n_\ell} \langle \sigma^2 \rangle + 2 \sum_{\alpha \neq \gamma}^{n_\ell} \langle \sigma \rangle^2 = n_\ell \langle \sigma^2 \rangle + 2 n_\ell (n_\ell - 1) \langle \sigma \rangle^2 \quad .$$

For n_ℓ a random variable,

$$\langle s_\ell^2 \rangle = \langle n_\ell \rangle \langle \sigma^2 \rangle + 2(\langle n_\ell^2 \rangle - \langle n_\ell \rangle) \langle \sigma \rangle^2 \quad ,$$

and

$$(\delta s_\ell)^2 = \langle \sigma_\ell^2 \rangle - \langle \sigma_\ell \rangle^2$$

$$= \langle n_\ell \rangle \langle \sigma^2 \rangle + 2(\langle n_\ell^2 \rangle - \langle n_\ell \rangle) \langle \sigma \rangle^2 - \langle n_\ell \rangle^2 \langle \sigma \rangle^2 \quad .$$

Rearranging terms and using $\langle n_\ell^2 \rangle = \langle n_\ell \rangle^2 + (\delta n_\ell)^2$ and $\langle \sigma^2 \rangle = \langle \sigma \rangle^2 + (\delta\sigma)^2$ leads to the result

$$\boxed{(\delta s_\ell)^2 = (\langle n_\ell \rangle^2 + 2(\delta n_\ell)^2 - \langle n_\ell \rangle) \langle \sigma \rangle^2 + \langle n_\ell \rangle (\delta\sigma)^2} \quad . \qquad (2)$$

If we assume Poisson statistics for the n_ℓ , then $(\delta n_\ell)^2 = \langle n_\ell \rangle$. Furthermore, we may write $(\delta\sigma)^2 = \chi^2 \langle \sigma \rangle^2$, where χ is a scale factor; for the resonant $\lambda/2$-dipole, $\chi = 4/3$. Then,

$$\boxed{(\delta s_\ell)_p^2 = [\langle n_\ell \rangle^2 + (1 + \chi^2) \langle n_\ell \rangle] \langle \sigma \rangle^2} \quad . \qquad (3)$$

Equation (3) indicates that for small values of $<n_\ell>$ the statistics of the single dipole, by virtue of χ^2, are important for calculating δs_ℓ. As $<n_\ell>$ increases, δs_ℓ approaches the average cross section which agrees with Rayleigh statistics.[1] Figure 1 indicates the ratio of $\delta s_\ell/<s_\ell>$ for several assumed of χ. If χ is of the order of one, Rayleigh statistics should be a good representation about the average if $<n_\ell>$ is greater than two or three.

III. Statistics For Several Types of Chaff.

Let us now consider that there are M types of dipoles to be distributed in space, and let

A_ℓ be the coherent return from the ℓ-th radar resolution cell,

$a_{i\alpha}$ be the coherent return from the α-th dipole, i-th type in the ℓ-th cell,

$r_{i\alpha}$ be the range to the α-th dipole, i-th type in the ℓ-th cell, and

$n_{\ell i}$ be the number of dipoles of the i-th type in the ℓ-th cell.

$$A_\ell = \sum_{i=1}^{M} \sum_{\alpha=1}^{n_{\ell i}} a_{i\alpha} e^{i2kr_{i\alpha}} .$$

$$s_\ell = A_\ell A_\ell^* = \sum_i^M \sum_j^M \sum_\alpha^{n_{\ell i}} \sum_\beta^{n_{\ell j}} a_{i\alpha} a_{j\beta}^* e^{i2k(r_{i\alpha} - r_{j\beta})} .$$

For fixed numbers $n_{\ell i}$,

$$\langle s_\ell \rangle = \int \prod_{m=1}^{M} \frac{\prod_{p=1}^{n_{\ell m}} d\vec{r}_{mp} \, d\Omega_{mp}}{(\Delta v)^{n_{\ell m}}} \sum_{i,j}^{M} \sum_\alpha^{n_{\ell i}} \sum_\beta^{n_{\ell j}} a_{i\alpha} a_{j\beta}^* e^{i2k(r_{i\alpha} - r_{j\beta})} .$$

But the volume integrals over $\vec{r}_{mp}$ are zero except for $i=j$, $\alpha=\beta$; thus,

$$\langle s_\ell \rangle = \int \prod_m^M \prod_p^{n_{\ell m}} d\Omega_{mp} \sum_i^M \sum_\alpha^{n_{\ell i}} a_{i\alpha} a_{i\alpha}^* = \sum_i^M \sum_\alpha^{n_{\ell i}} \langle \sigma_i \rangle$$

$$\langle s_\ell \rangle = \sum_{i=1}^{M} n_{\ell i} \langle \sigma_i \rangle .$$

If the $n_{\ell i}$ are random, independent variables,

$$\boxed{\langle s_\ell \rangle = \sum_{i=1}^{M} \langle n_{\ell i} \rangle \langle \sigma_i \rangle} . \qquad (4)$$

Again, to determine the standard deviation of s_ℓ, we must calculate

$$<s_\ell^2> = \int \prod_{m=1}^{M} \frac{\prod_{p=1}^{n_{\ell m}} d\vec{r}_{mp} d\Omega_{mp}}{(\Delta v)^{n_{\ell m}}} \sum_{g,h,i,j}^{M} \sum_{\alpha}^{n_{\ell g}} \sum_{\beta}^{n_{\ell n}} \sum_{\gamma}^{n_{\ell i}} \sum_{\delta}^{n_{\ell j}} a_{g\alpha} a_{h\beta} a^*_{i\gamma} a^*_{j\delta} e^{i2k(r_{g\alpha}+r_{h\beta}-r_{i\gamma}-r_{j\delta})} .$$

The volume integrals are only non-zero for the cases:

1. $g = h$, which implies $i = j = g = h$
   a) $\alpha = \beta = \gamma = \delta$
   b) $\alpha \neq \beta$
      1) $\alpha = \gamma$, $\beta = \delta$
      2) $\alpha = \delta$, $\beta = \gamma$
2. $g \neq h$
   a) $g = i$ and $\alpha = \gamma$ while $h = j$ and $\beta = \delta$
   b) $g = j$ and $\alpha = \delta$ while $h = i$ and $\beta = \gamma$.

Thus,

$$<s_\ell^2> = \int \prod_{m=1}^{M} \prod_{p=1}^{n_{\ell m}} d\Omega_{mp} \Big[\sum_{g=1}^{M} \sum_{\alpha=1}^{n_{\ell g}} (a_{g\alpha} a^*_{g\alpha})^2 + 2 \sum_{g=1}^{M} \sum_{\alpha}^{n_{g\alpha}} \sum_{\neq \beta}^{n_{g\beta}} (a_{g\alpha} a^*_{g\alpha})(a_{g\beta} a^*_{g\beta})$$

$$+ 2 \sum^{M}_{g} \sum^{M}_{\neq h} \sum_{\alpha=1}^{n_{\ell g}} \sum_{\beta=1}^{n_{\ell h}} (a_{g\alpha} a^*_{g\alpha})(a_{h\beta} a^*_{h\beta}) \Big]$$

$$= \sum_{g=1}^{M} \sum_{\alpha=1}^{n_{\ell g}} <\sigma_g^2> + 2 \sum_{g=1}^{M} \sum_{\alpha}^{n_{g\alpha}} \sum_{\neq \beta}^{n_{g\beta}} <\sigma_g>^2 + 2 \sum^{M} \sum^{M}_{g \neq h} \sum_{\alpha}^{n_{\ell g}} \sum_{\beta}^{n_{\ell h}} <\sigma_g> <\sigma_h> .$$

Changing the summation variables g, h to i, j we finally determine

$$<s_\ell^2> = \sum_{i=1}^{M} n_{\ell i} <\sigma_i^2> + 2 \sum_{i=1}^{M} n_{\ell i}(n_{\ell i} - 1)<\sigma_i>^2 + 2 \sum^{M}_{i} \sum^{M}_{\neq j} n_{\ell i} n_{\ell j} <\sigma_i> <\sigma_j> .$$

If the $n_{\ell i}$ are now considered random independent variables,

$$<s_\ell^2> = \sum_{i=1}^{M} <n_{\ell i}><\sigma_i^2> + 2\sum_{i=1}^{M}(<n_{\ell i}^2> - <n_{\ell i}>)<\sigma_i>^2 + 2\sum_{i \neq j}^{M}\sum^{M} <n_{\ell i}><n_{\ell j}><\sigma_i><\sigma_j> ,$$

and

$$(\delta s_\ell)^2 = <s_\ell^2> - <s_\ell>^2$$

$$= \sum_{i=1}^{M} <n_{\ell i}><\sigma_i^2> + 2\sum_{i=1}^{M} (<n_{\ell i}^2> - <n_{\ell i}>)<\sigma_i>^2 + 2\sum_{i \neq j}^{M}\sum^{M} <n_{\ell i}><n_{\ell j}><\sigma_i><\sigma_j>$$

$$- \sum_{i}^{M}\sum_{j}^{M} <n_{\ell i}><n_{\ell j}><\sigma_i><\sigma_j> .$$

Rearranging terms finally leads to the result

$$\boxed{(\delta s_\ell)^2 = \sum_{i=1}^{M} [(<n_{\ell i}>^2 + 2(\delta n_{\ell i})^2 - <n_{\ell i}>)<\sigma_i>^2 + <n_{\ell i}> (\delta \sigma_i)^2] + \sum_{i \neq j}^{M}\sum^{M} <n_{\ell i}><n_{\ell j}><\sigma_i><\sigma_j> .} \qquad (5)$$

If we let $(\delta \sigma_i)^2 = \chi_i^2 <\sigma_i>^2$, and again assume Poisson statistics for each $n_{\ell i}$,

$$(\delta n_{\ell i})^2 = <n_{\ell i}> ,$$

and

$$\boxed{(\delta s_\ell)^2 = \sum_{i=1}^{M} (<n_{\ell i}>^2 + (1 + \chi_i^2) <n_{\ell i}>)<\sigma_i>^2 + \sum_{i \neq j}^{M}\sum^{M} <n_{\ell i}><n_{\ell j}><\sigma_i><\sigma_j> .} \qquad (6)$$

Note that the case $M = 1$, i. e., only one type of dipole, reduces to the result found previously. The cross terms provide for the Rayleigh limit.

As one final calculation, let us assume that by virtue of the radar resolution, the s_ℓ are statistically independent. (This assumption would

by physically violated if there were large range Doppler coupling and a wide spread in local dipole velocity. Further violation would be mathematically introduced if the data were analyzed by overlapping resolution volumes to measure the s_ℓ .)

To calculate the incoherent radar return from N resolution cells, one defines

$$S = \sum_{\ell=1}^{N} s_\ell \quad .$$

Now we find

$$\langle S \rangle = \sum_{\ell=1}^{N} \langle s_\ell \rangle \quad , \tag{7}$$

and

$$\langle S^2 \rangle = \langle \sum_{k=1}^{N}\sum_{\ell=1}^{N} s_k s_\ell \rangle = \sum_{\ell=1}^{N} \langle s_\ell^{\,2} \rangle + \sum_{k\neq\ell}^{N} \sum^{N} \langle s_k \rangle \langle s_\ell \rangle \quad .$$

$$(\delta S)^2 = \langle S^2 \rangle - \langle S \rangle^2 = \sum_{\ell=1}^{N} \langle s_\ell^{\,2} \rangle + \sum_{k\neq\ell}^{N}\sum^{N} \langle s_k \rangle \langle s_\ell \rangle - \sum_{k}^{N}\sum_{\ell}^{N} \langle s_k \rangle \langle s_\ell \rangle \quad .$$

$$(\delta S)^2 = \sum_{\ell=1}^{N} \langle s_\ell^{\,2} \rangle - \langle s_\ell \rangle^2 = \sum_{\ell=1}^{N} (\delta s_\ell)^2 \quad . \tag{8}$$

If the $\langle s_\ell \rangle$ and the δs_ℓ were all roughly equal to some $\langle s \rangle$ and $\delta s \simeq \langle s \rangle^2$ respectively for these N cells, the final result is

$$\langle S \rangle \simeq N \langle s \rangle$$

$$(\delta S)^2 \simeq N (\delta s)^2 \simeq N \langle s \rangle^2 \tag{9}$$

and

$$(\delta S)/\langle S \rangle \simeq \frac{1}{\sqrt{N}}$$

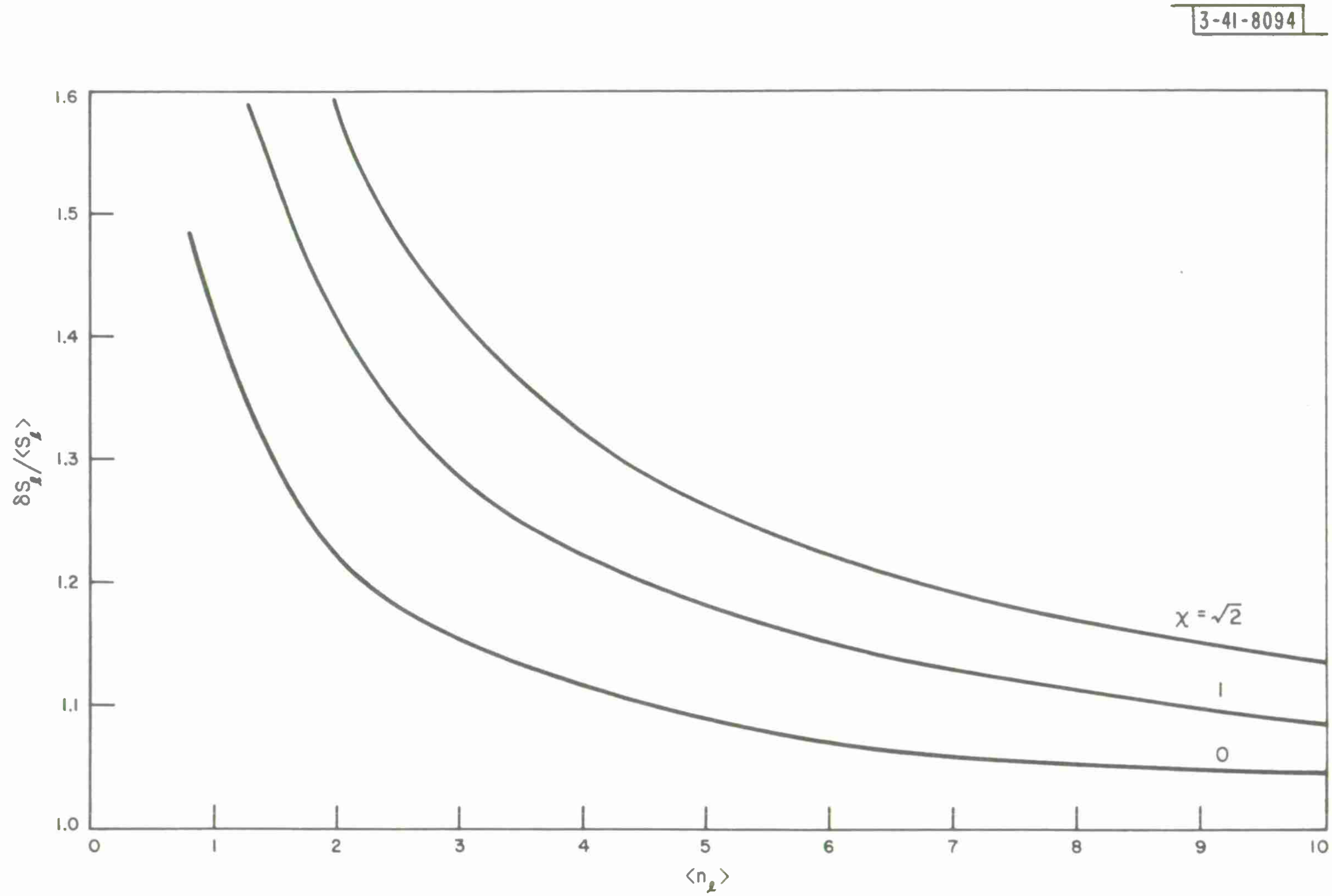

Figure 1. Ratio of the standard deviation to the average cross section of a resolution volume for several values of the standard deviation of the single dipole cross section.

REFERENCES

1. S. L. Borison, "Probability Density for the Radar Cross Section of One or More Randomly-Oriented Dipoles, Lincoln Laboratory, M.I.T. Group Report 1964-33 (22 June 1964) (U).

2. For a calculation of the average cross section of a thin wire, see for example, Van Vleck et al., "Theory of Radar Reflection from Wires or Thin Metallic Strips," J. Appl. Phys. 18, pp. 274-294 (1947).

3. J. Rheinstein, Monte Carlo computation of the probability density for the radar cross section of thin wires 2.5 to 4.6 wavelengths long. (Not generally available)

DISTRIBUTION LIST

Division 4

J. Freedman

Group 41

S. L. Borison

D. L. Clark

E. M. Hofstetter

J. B. Resnick

C. B. Slade

D. M. Towle

Group 45

D. F. DeLong

M. W. Dill

J. P. Perry

DOCUMENT CONTROL DATA - DD 1473

1. ORIGINATING ACTIVITY Lincoln Laboratory, M. I. T.	2a. REPORT CLASSIFICATION Unclassified 2b. DOWNGRADING GROUP
3. REPORT TITLE Statistics of the Radar Cross Section of a Volume of Chaff	
4. TYPE OF REPORT AND INCLUSIVE DATES Group Report	
5. AUTHOR(S) (Last name first) Borison, S. L.	

6. REPORT DATE 10 February 1965	7a. NO. OF PAGES 15	7b. NO. OF REFS. 3
8a. CONTRACT NO. AF 19(628)-500	9a. ORIGINATOR'S REPORT NO. Group Report 1965-10	
8b. ORDER NO. ARPA Order 498	9b. OTHER REPORT NO(S). ESD-TDR-65-46	
10. AVAILABILITY OR LIMITATION NOTICES		
11. SUPPLEMENTARY NOTES	12. SPONSORING ACTIVITY Advanced Research Projects Agency	

13. ABSTRACT

The average and standard deviation of the radar cross section of chaff is computed under the assumptions that dipoles are randomly oriented and randomly distributed within a radar resolution volume. For a single type of chaff, the standard deviation approaches the average value as the average number of dipoles increases. When the number of dipoles per resolution volume is small, the statistics of the single dipole cross section are important. The results are then generalized to the case of several types of dipoles distributed in space. The average cross section is simply the sum of the average cross sections for each type; however, the standard deviation involves additional terms which are not small. These terms are just sufficient to again provide the Rayleigh limit $\delta s/\langle s\rangle \to 1$ as the number of dipoles increases.

14. KEY WORDS

radar cross sections
chaff
dipoles